Perturbed Markov Chains, Application to Distributed Optimization

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Distributed Optimization

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N is the number of dimensions (agents)
$\{0, A\}$ is the action space of each agent (w.n.l.g.).

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x is a local optimum if $\forall i$, $F(x) = \max_{\alpha \in \{0,A\}} F(\alpha, x_{-i})$.

Assumption (A)

We assume that for all i and for all x,

$$\operatorname*{argmax}_{\alpha \in \{0,A\}} F(\alpha, x_{-i}) \quad \text{is unique.}$$

Example in dimension $N = 2$

1	3	1	0	4	2	1	0
4	1	9	0	0	3	2	0
5	1	3	3	4	1	1	2
7	3	1	4	6	2	1	1
0	4	0	3	0	2	0	3
2	3	0	0	5	4	1	1
5	0	1	5	0	1	6	0
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1. Pick one agent i (with a given distribution over all agents)
2. Agent i chooses the action that maximizes F
3. Go back to 1.

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Convergence to Local Optima

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Proof. Each time one coordinate is changed, the value increases (so it must converge to a local optimum).

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Randomized Algorithm (RA)

1. Pick one set S of players (with a given distribution ρ).
2. Each agent i in S chooses action $Q_i(x)$
3. Go back to 1.

Randomized Algorithm (II)

The evolution of the state x is Markovian. The transition matrix has two parts: first choose the revision set S, then choose the new action for each agent in S.

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$$P_{x,y} = \sum_{S \supseteq \mathrm{Diff}(x,y)} \rho(S) \prod_{i \in S} \frac{e^{\theta F(y_i, x_{-i})}}{\sum_{\alpha \in A} e^{\theta F(\alpha, x_{-i})}}.$$

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When $\theta \to \infty$, (RA) $\to$ (GA),
however $\pi_x(\theta) \not\to \pi_x(\infty)$, (the stable states of (GA)), but selects a subset.

Stochastically stable states for (RA)

Theorem (Caracterization of stochastic stable states)

State x is stochastically stable if and only if the order of its minimal in-tree is the smallest, among all in-trees.

proof. Let us use the caracterization of stochastically stable states.
$\varepsilon = e^{-\theta}$

$$P_{x,y} = \sum_{S \supseteq \mathrm{Diff}(x,y)} \rho(S) \prod_{k \in S} \frac{\varepsilon^{-F(y_k, x_{-k})}}{\sum_{\alpha \in A(k)} \varepsilon^{-F(\alpha, x_{-k})}}.$$

which can be written $P_{x,y} = c_{x,y}\varepsilon^{q_{x,y}} + o(\varepsilon^{q_{x,y}})$,

$$q_{x,y} \stackrel{\mathrm{def}}{=} \min_{S \supseteq \mathrm{Diff}(x,y) \cap \mathcal{S}(\rho)} \left(\sum_{k \in S} \left(\max_{\alpha \in \mathcal{A}_k} F(\alpha, x_{-k}) - F(y_k, x_{-k}) \right) \right).$$

Stochastically stable states for (RA) (II)

By using the Markov chain tree theorem, the order q_x of π_x w.r.t. ε is

$$q_x \stackrel{\text{def}}{=} \min_{T \in \mathcal{T}_x} \sum_{(y,z) \in T} q_{y,z}$$

Therefore, the only components in π that do not go to 0 when ε goes to 0 are those with the smallest order:

$$(\lim_{\theta \to \infty} \pi_x > 0) \Leftrightarrow (q_x = \min_{y \in \mathcal{A}} q_y). \qquad (1)$$

Convergence to Local Optima

Theorem (Convergence to global optima for asynchronous revisions)

If the revision family is separable, then the stochastically stable states are local optima.

proof. $q_{x,y} \geqslant 0$. $q_{x,y} = 0 \Leftrightarrow \exists S \in \mathrm{Diff}(x,y) \cap \mathcal{S}(\rho)$ s.t. $y = \mathrm{BR}_S(x)$.
If x is not a local optimum, then separability implies $\exists (S_n)_{0 \leqslant n < H}$ of sets of players in $\mathcal{S}(\rho)$ and states $(X_n)_{0 \leqslant n \leqslant H}$ such that

$$\begin{aligned} X_0 &= x \\ X_{n+1} &= \mathrm{BR}_{S_n}(X_n), \quad \forall\, 0 \leqslant n < H, \end{aligned}$$

and X_H is a local optimum.
This constructs a path with order 0 from x to X_H. Let T_x^* be a tree with minimal order, routed in x. From T_x^*, construct a tree routed in X_H by adding the path from x to X_H and removing the arc in T_x^* starting in X_H. This arc's order > 0. The new tree's order is strictly smaller that T_x^*, so x cannot achieve the minimum in (1).

Convergence to Global Optima

Theorem (Convergence to global optima for asynchronous revisions)

If the revision family is only made of all the singletons, then the only stochastically stable states are the global optima.

proof. Under uniform selction, the Markov chain (X_n) is reversible and the stationary probability is explicitly known: for all profiles x,

$$\pi_x \propto e^{\theta F(x)}.$$

When $\theta \to \infty$, the total stationary probability of the profiles with optimal potential will go to one.

Example 1: 2 agents, no convergence

$F =$

$1\backslash 2$	a	b
a	1	0.5
b	0	1

Revision set: $\{1, 2\}$

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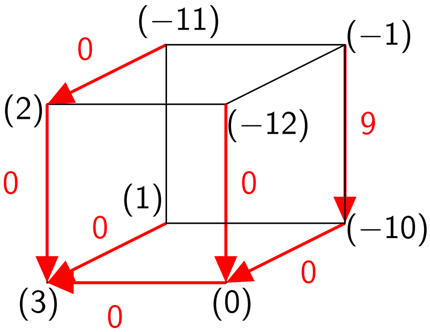
$$\pi(((a, a), (a, b), (b, a), (b, b)) \to (1/4, 1/4, 1/4, 1/4).$$

Example 2: 3 agents, convergence to LO



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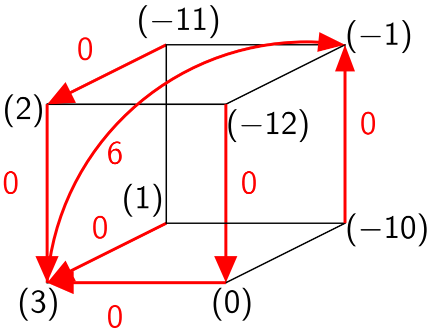

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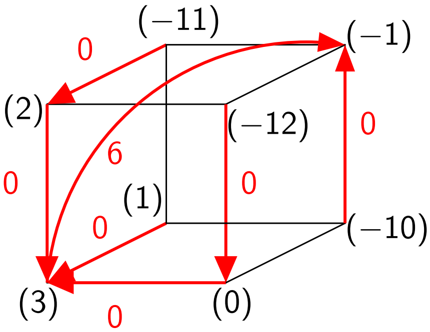

Revision set: $\{1\}, \{2\}, \{3\}, \{1, 2, 3\}$.

Unique stable state: $(1, 1, 1)$ (not global optimum).

Examples 3: 2 agents, convergence to LO

$F =$

1\2	a	b	c
a	11	0	5
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Unique stable state: (b, b), not global optimum.

Example 4: 2 agents, no convergence

$F =$

1\2	a	b
a	1	1
b	1	0

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$$F = \begin{array}{|c||c|c|} \hline 1\backslash 2 & a & b \\ \hline a & 1 & 1 \\ b & 1 & 0 \\ \hline \end{array}$$

Revision set $\{1,\},\{2\},\{1,2\}$.

$\pi((a,a),(a,b),(b,a),(b,b)) \to (36/79, 20/79, 20/79, 3/79)$

Separable Families

Let $\mathcal{R}$ be a family of sets and consider the following elimination process:

As long as there is a singleton (say $\{k\}$) in $\mathcal{R}$, remove k from all sets in $\mathcal{R}$.

$\mathcal{R}$ is *separable* if the elimination process reduces $\mathcal{R}$ to the empty set.

Example:
$\mathcal{R}_1 = \{1\}, \{1,2,3\}, \{2,4\}, \{1,4\}$ is separable
but
$\mathcal{R}_2 = \{1\}, \{1,2,3\}, \{2,4\}, \{3,4\}$ is not separable

$\mathcal{R}_3 =$ all the sets obtained when each agent i decides to play with probability p_i is separable (and fully distributed).

Separability and Convergence to Local Optima

Theorem

The algorithm GA converges to a local optimum for all functions F satisfying (A) if and only if the revision set is separable.

Proof.

1) By contradiction.

If $\mathcal{R}$ is separable, and GA does not converge to a local optimum, let x be the state with maximal value visited by GA.

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From x, let us select agents in the order of the elimination for separability (this happens with a positive probability).
The value increases (impossible) or x is a local optimum (impossible).

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The rest holds by induction on N.