Oracle skipping and applications to Jackson networks

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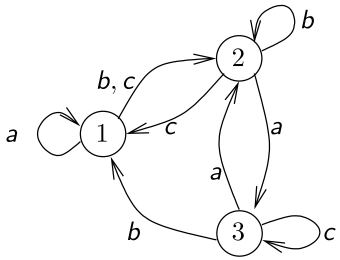
Markov automaton

Markov automaton

$\mathcal{A} = (\mathcal{S}, A, D, \cdot)$, where

- $\mathcal{S}$ is a finite *state space*;
- A is a finite alphabet (the set of *events*);
- D is a probability distribution over A;
- $\cdot : \mathcal{S} \times A \to \mathcal{S}$; $(s, a) \mapsto s \cdot a$ is an action by the letters of A on the states of $\mathcal{S}$.

$u[i]$: prefix of u of length i.
For $S \subseteq \mathcal{S}$, $S \cdot a = \{s \cdot a \mid s \in S\}$.
Bounding chain: $S \cdot a \subseteq S \circ a$



$$D(a) = D(b) = D(c) = 1/3$$

Markov chain generated by $\mathcal{A}$: let $s \in \mathcal{S}$ and $u \sim D^{\otimes \mathbb{N}}$.

$$X_n(s) = s \cdot u[i].$$

Coupling in Markov automata

Grand coupling

$$\mathcal{X} = (X(s))_{s\in\mathcal{S}} \qquad \mathcal{X}_i = \mathcal{S} \cdot u[i].$$

Coupling word

u such that $|\mathcal{S} \cdot u| = 1$.

Example : *bb*

If there exists a coupling word, then the algorithm terminates with probability 1.

Coupling from the past

Algorithm 1: Coupling from the past

for $s \in \mathcal{S}$ **do** $S(s) \leftarrow s$ **repeat**
    Draw $a \sim D$;
    for $s \in \mathcal{S}$ **do** $T(s) \leftarrow S(s \cdot a)$;
    $S \leftarrow T$;
until $|S(\mathcal{S})| = 1$;
return *the element of* $S(\mathcal{S})$

- τ_b is the backward coupling time (number of steps)
- If τ is the (forward) coupling of the chain, then $\tau_{mix} \leq \mathbb{E}[\tau] = \mathbb{E}[\tau_b]$

$$t_{mix} = \min\{i \mid \max_{x \in \mathcal{S}} ||\rho_i(x) - \pi||_{TV} \leq 1/4\}$$

with $||\rho - \pi||_{TV} = \max_{B \subseteq \mathcal{S}} |\rho(B) - \pi(B)|$.

Active and passive events

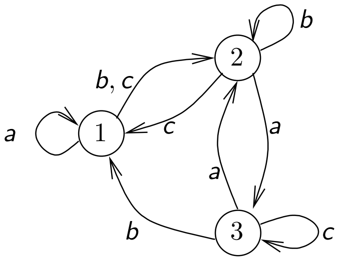
Let $B \subseteq \mathcal{S}$ and $a \in A$.

Active event

The event a is *active* for B if $B \circ a \neq B$.

Passive event

The event a is *passive* for B if $B \circ a = B$.



S	active	passive
$\{1,2,3\}$	b	a, c
$\{1,2\}$	a, b	c
$\{2,3\}$	b, c	a
$\{1\}$	b, c	a
$\{2\}$	a, c	b
$\{3\}$	a, b	c

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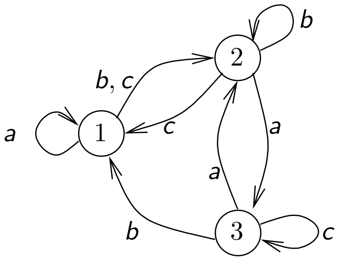

New distribution D_i at step i

$P_{D_i}(u_i = a) = P_D(u_i = a \mid a \text{ is active })$

In state $\{1, 2\}$, $P(a) = 1/2$, $P(b) = 1/2$ and $P(c) = 0$.

Hard on CFTP

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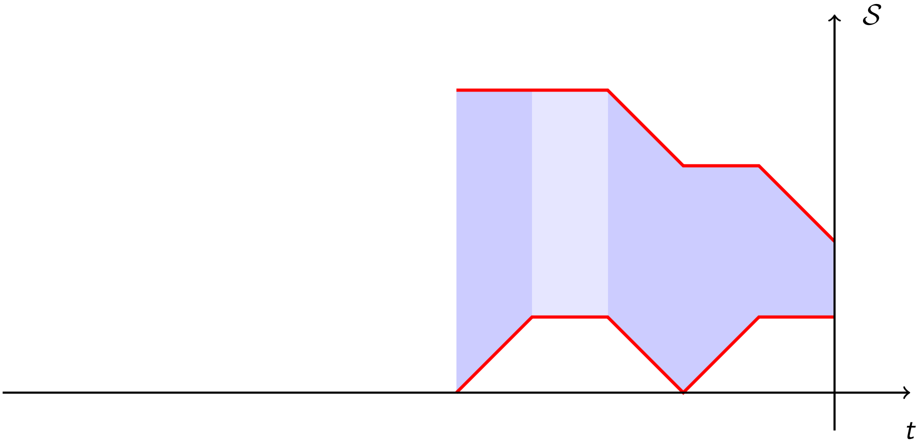

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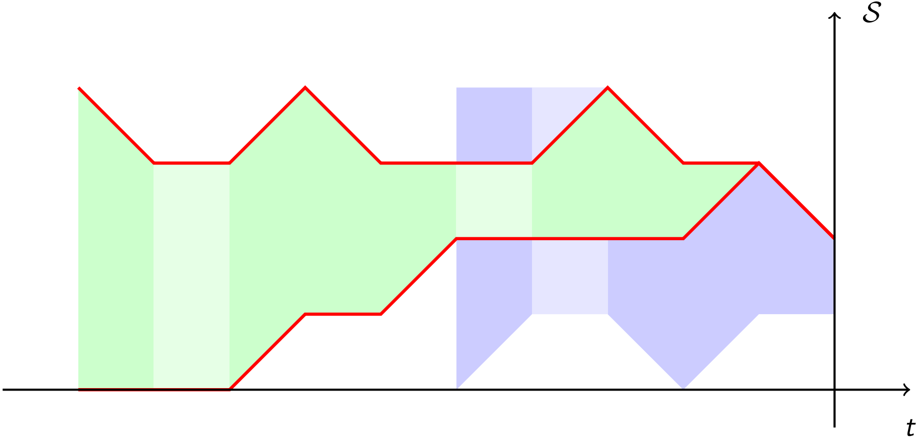

Special symbol $\sharp$

Let $A_\sharp = A \cup \{\sharp\}$.

- The new symbol $\sharp$ has no effect: $\forall B \subseteq \mathcal{S}$, $B \cdot \sharp = B$.
- If D is a distribution over A and $p \in (0,1)$, then D_p is a distribution over $A_\sharp$ such that
  - $D_p(\sharp) = p$
  - and $D_p(a) = (1-p)D(a)$.

$\sharp$ is always considered as active:

- $Act_B = \{a \in A \mid B \circ a \neq B\} \cup \{\sharp\}$
- $Pas_B = \{a \in A \mid B \circ a = B\}$

Collapsing a word = removing its inactive letters

Let $u \in A^n$, $n \in \mathbb{N} \cup \{\infty\}$ and $Act_i = Act_{\mathcal{S} \circ u[i]}$.
$c(u) = u_{\phi(1)} \cdot u_{\phi(2)} \cdots u_{\phi(\ell)}$, where

- $\phi(i) = \min\{j > \phi(i-1) \mid u_j \in Act_{\phi(i-1)}\}$ and $\phi(0) = 0$;
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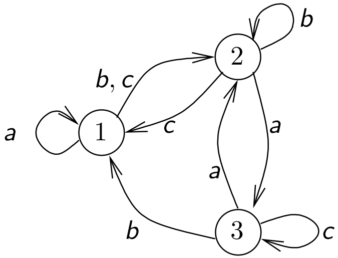

$u = aacbcacaacacb$
$c(u) =$

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Let $u \in A^n$, $n \in \mathbb{N} \cup \{\infty\}$ and $Act_i = Act^B{}_{\mathcal{S} \circ u[i]}$.
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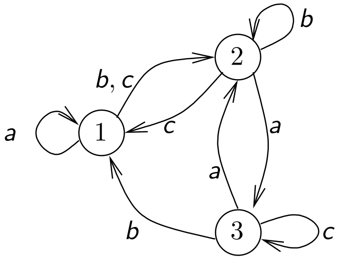

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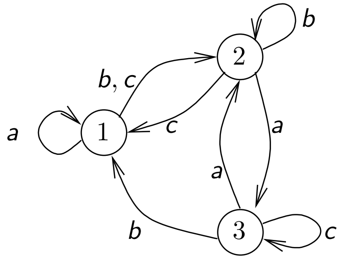

Lemma

$$c(u \cdot v) = c(u) \cdot c^{\mathcal{S} \circ u}(v)$$

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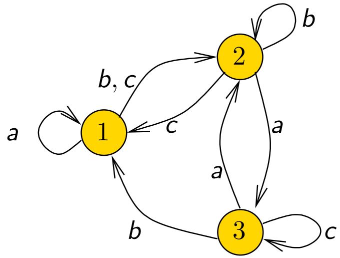

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$u = aacbcacaacacb$

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$Act = \{b\}$

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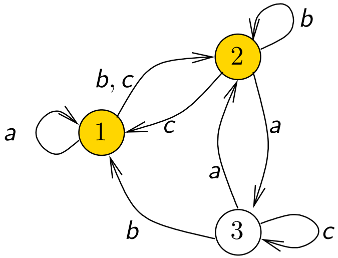

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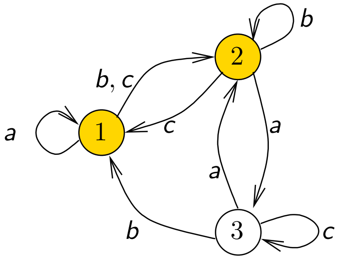

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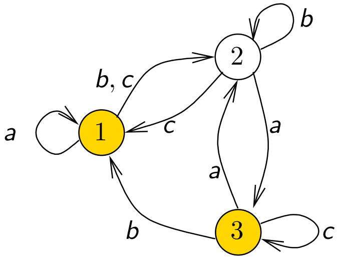

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$u = aacbcacaacacb$
$c(u) = ba$
$Act = \{a, b, c\}$

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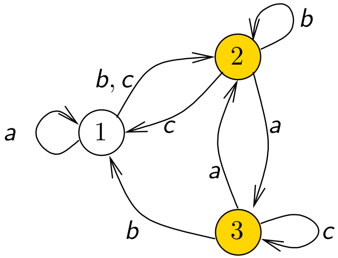

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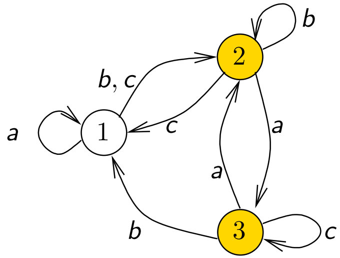

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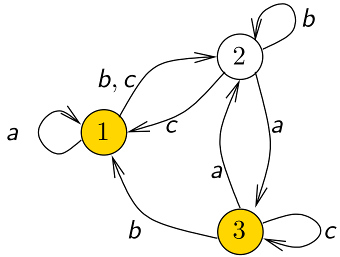

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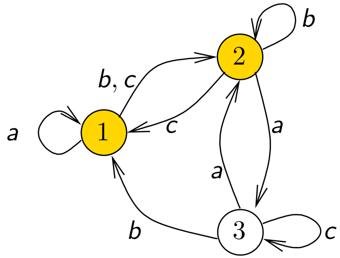

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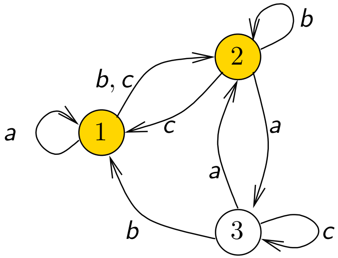

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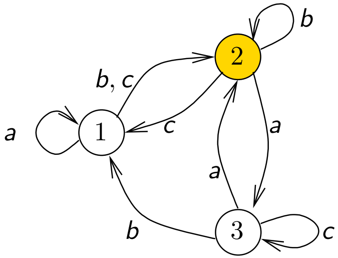

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$Act = \{a, c\}$

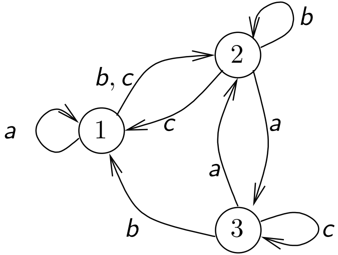
p-expansion of a word

Let $v = v_1 \cdots v_\ell \in A^\ell$. The p-expansion of v is

$$e_p(v) = w_0 v_1 w_1 \cdots w_{\ell-1} v_\ell$$

where $w_i \in A^*$ and

- $|w_i| \sim \mathcal{G}eo(p_{Act_i}) - 1$
- the letters of w_i are i.i.d according to the distribution of the passive letters D_{Pas_i}



$u = baccab$

$e_p(u) =$

p-expansion of a word

Let $v = v_1 \cdots v_\ell \in A^\ell$. The p-expansion of v from B is

$$e^{B}{}_{p}(v) = w_0 v_1 w_1 \cdots w_{\ell-1} v_\ell$$

where $w_i \in A^*$ and

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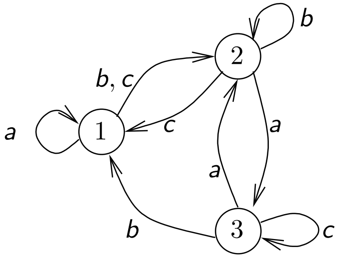

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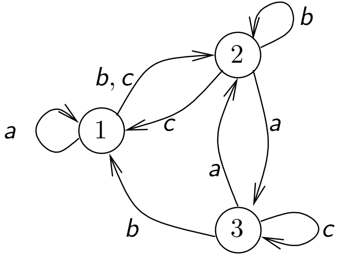

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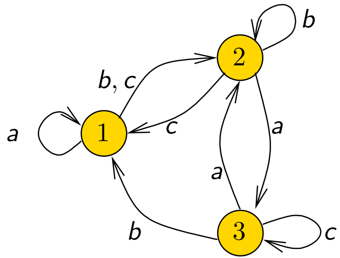

$u = baccab$

$e_p(u) = ca$

$Act = \{b\}$

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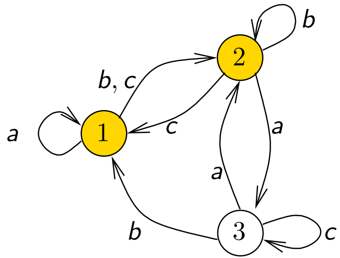

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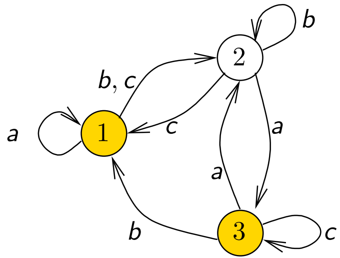

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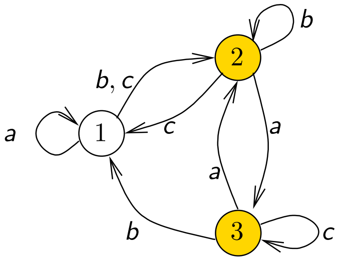

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$Act = \{b, c\}$

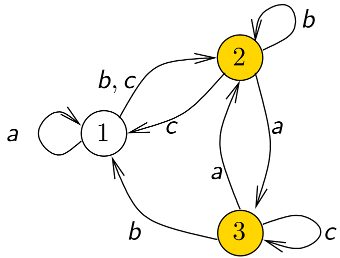
p-expansion of a word

Let $v = v_1 \cdots v_\ell \in A^\ell$. The p-expansion of v is

$$e_p(v) = w_0 v_1 w_1 \cdots w_{\ell-1} v_\ell$$

where $w_i \in A^*$ and

- $|w_i| \sim \mathcal{G}eo(p_{Act_i}) - 1$
- the letters of w_i are i.i.d according to the distribution of the passive letters D_{Pas_i}



$u = baccab$

$e_p(u) = cabacaa$

$Act = \{b, c\}$

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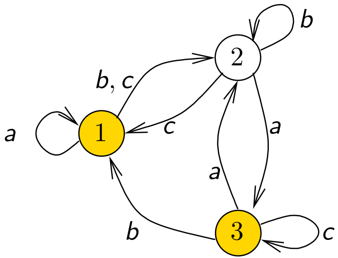

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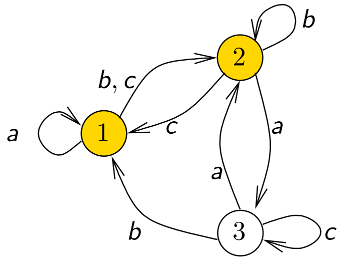

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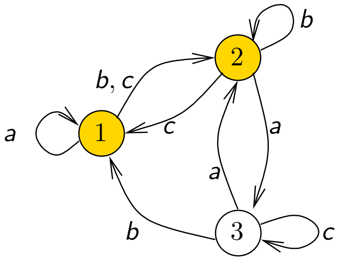

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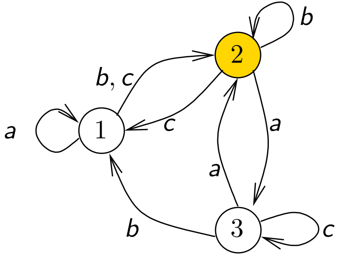

$u = baccab$

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$Act = \{a, c\}$

Expansion of a collapsed word

Lemma

Let $u \in (A^\sharp)^{\mathbb{N}}$ *such that* $u \sim D_p^{\otimes \mathbb{N}}$. *Then* $e_p(c(u)) \sim D_p^{\otimes \mathbb{N}}$.

Applying e_p to a collapsed word corresponds to what the word *could have been* before it was collapsed. It does not change the bounding state reached at the end.

Lemma

Let $u \in (A^\sharp)^{\mathbb{N}}$ *such that* $u \sim D_p^{\otimes \mathbb{N}}$, *and* $u^\sharp$ *be the word truncated after the first occurrence of* $\sharp$. *Call* G_p *the distribution of* $u^\sharp$. *Then*

$$e_p(c(u^\sharp)) \sim G_p.$$

$\sharp$ is always an active letter, so the occurrences of $\sharp$ are preserved in u and $e_p(c(u))$

$\mathcal{G}$-expansion of a word

G_p : distribution of a word according to $D_p^{\otimes}$ truncated after the first occurrence of $\sharp$.

Let $u = u^n \cdots u^2 u^1$ a word such that

- the u_m are mutually independent
- $u_m \sim G_{2^{-m}}$.

We denote by $\mathcal{G}_n$ the distribution of such a word.

- A word distributed according $\mathcal{G}_n$ has exactly n symbols $\sharp$ and ends with $\sharp$.
- It can be decomposed in a unique way into $u^1, \ldots, u^n$ respectively distributed according $G_{2^{-1}}, \ldots, G_{2^{-n}}$.

$\mathcal{G}$-expansion of a word

$\mathcal{G}$-expanded word: Let $v = u^n \cdots u^1 \sim \mathcal{G}_n$.

$$e_{\mathcal{G}}(v) = e_{2^{-n}}^{B_n}(u^n) \cdots e_{2^{-m}}^{B_m}(u^m) \cdots e_{1/2}^{B_1}(u^1),$$

with $B_m = \mathcal{S} \circ u_n \cdots u_{m+1}$.

Lemma

$$u \sim \mathcal{G}_n \Rightarrow e_{\mathcal{G}}(c(u)) \sim \mathcal{G}_n.$$

$$c(u \cdot v) = c(u) \cdot c^{\mathcal{S} \circ u}(v)$$

so

$$\begin{aligned} e_{\mathcal{G}}(c(u)) &= e_{\mathcal{G}}(c^{B_n}(u^n) \cdots c^{B_m}(u^m) \cdots c^{B_1}(u_1)) \\ &= e_{2^{-n}}^{B_n}(c^{B_n}(u^n)) \cdots e_{2^{-m}}^{B_m}(c^{B_m}(u^m)) \cdots e_{1/2}^{B_1}(c^{B_1}(u^1)). \end{aligned}$$

Main theorem

We define the words $w^o = \epsilon$ and $w^{n+1} \sim c(u^{n+1} e_{\mathcal{G}}(w^n))$.
For all n, $w^n \sim \mathcal{G}_n$.

Theorem

If a Markov automaton $\mathcal{A}$ is coupling, then

$$P(\exists n \in \mathbb{N} \mid |\mathcal{S} \circ w^n| = 1) = 1$$

and

$$\mathbb{E}[\min\{n \in \mathbb{N} \mid |\mathcal{S} \circ w^n| = 1\}] < \infty.$$

Moreover, for any $n \in \mathbb{N}$ such that $|\mathcal{S} \circ w^n| = 1$, then the unique element of $\mathcal{S} \circ w^n$ is distributed according to the stationary distribution π of $\mathcal{A}$.

Algorithm

Algorithm 2: CFTP with oracle skipping

$n \leftarrow 0$; $w \leftarrow \epsilon$;
repeat
    $n \leftarrow n+1$; $m \leftarrow n-1$;
    generate $u \sim c(G_{2^{-n}})$;
    $Act^{old} \leftarrow \mathcal{S}$; $Act \leftarrow Act_{\mathcal{S} \circ u^n}$;
    while $w \neq \epsilon$ **do**
        Draw $a \sim D_{2^{-m}}(Act \cup Act^{old})$;
        if $a \in Act^{old}$ **then**
            $w \leftarrow w_1^{-1} \cdot w$;
            if $w_1 \in Act$ **then**
                $u \leftarrow uw_1$;
                if $w_1 = \sharp$ **then** $m \leftarrow m-1$
        else $u \leftarrow ua$;
    $w \leftarrow u$
until $|\mathcal{S} \circ w| = 1$;

Update Act^{old} and Act each time w or u are updated.

Difficulty

Draw a such that a is active for either u or w.

Proof

With $w^o = \epsilon$ and $w^{n+1} \sim c(u^{n+1} e_{\mathcal{G}}(w^n))$.

1. **Convergence:** There exists a coupling word u with $|u| = k$.

$$P(u^i \text{ contains } u) \geq \frac{1}{2^{|u|}} P_u$$

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$$\mathcal{S} \circ w^{n+1} \subseteq \mathcal{S} \circ w^n$$

$$\begin{aligned}
\mathcal{S} \circ w^{k+1} &= \mathcal{S} \circ c(u^{k+1} \cdot e_p(w^k)) \\
&= \mathcal{S} \circ u^{k+1} \circ e_p(w^k) \\
&\subseteq \mathcal{S} \circ e_p(w^k) \\
&= \mathcal{S} \circ w^k
\end{aligned}$$

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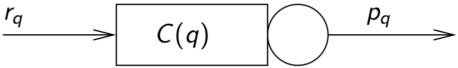
3. **Convergence to the stationary distribution:** same as in the classical proof

Lower bound on the mixing time of a Jackson network



Theorem

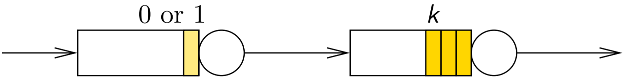
Let q be a queue. The mixing time t_{mix} of the automaton satisfy

$$t_{mix} \geq \frac{C(q)}{8 \max(p_q, r_q)},$$

where $p_q = \sum_{q'} D(q, q')$ and $r_q = \sum_{q'} D_{q',q}$.

Coupling in a Jackson network

A queue that has coupled can uncouple.



Proposition

In an acyclic Jackson network, if a queue couples when all its ancestors have coupled, it cannot uncouple.

Theorem (Coupling time of a single M/M/1/C queue)

The expected number of events it takes a M/M/1/C queue to couple is at most $\frac{C+C^2}{2}$.

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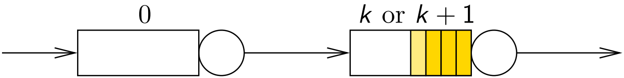

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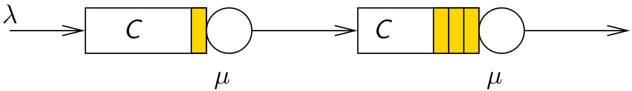
Coupling time of the first queue



$\rho = \lambda/\mu$

Let τ_1 be the coupling time of the first queue (no skipping)

Proposition (Coupling time of the first queue)

$\mathbb{E}[\tau_1] = C + C^2$

Let Y be the embedded chain with only the arrivals and services of the first queue.

$$\mathbb{E}[\tau_1] = \frac{\lambda+\mu}{\lambda+2\mu}\mathbb{E}[\tau_Y] = \frac{\rho+2}{\rho+1}\frac{C+C^2}{2} \leq C + C^2.$$

Second queue: skipping of the passive arrivals

$\tau_{2|1}$ coupling time of 2 from τ_1

- n_i^0 number of arrivals up to time i;
- n_i^q number of services of queue q up to time i;

In the first queue: x_0 state at τ_1

$$x_i = x_0 + n_i^0 - n_i^1 \leq C \quad \text{and} \quad n^0_{\tau_{2|1}} \leq n^1_{\tau_{2|1}} + C$$

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Coupling time of the tandem

$$E[\tau] = \mathbb{E}[\tau_1] + \mathbb{E}[\tau_{2|1}] \leq 4C + 3C^2.$$

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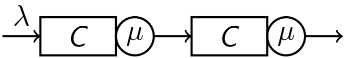
$$E[\tau] = \mathbb{E}[\tau_1] + \mathbb{E}[\tau_{2|1}] \leq 4C + 3C^2.$$

Without skipping, we have $\mathbb{E}[\tau] = O(C^2\rho)$.

Performance



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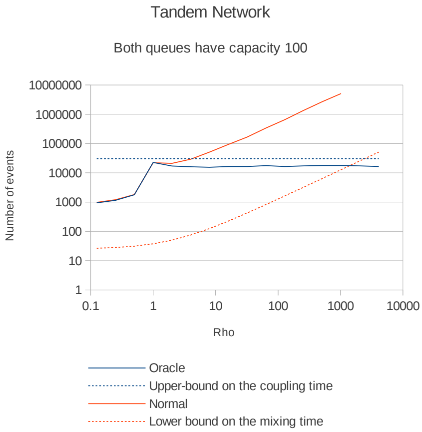

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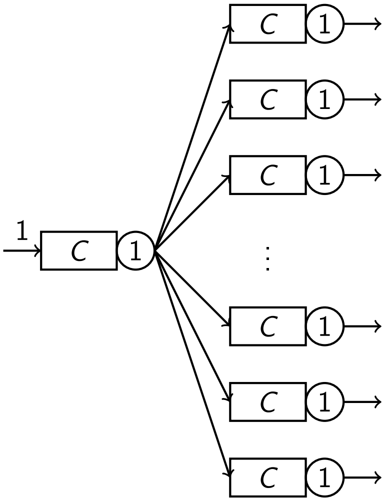

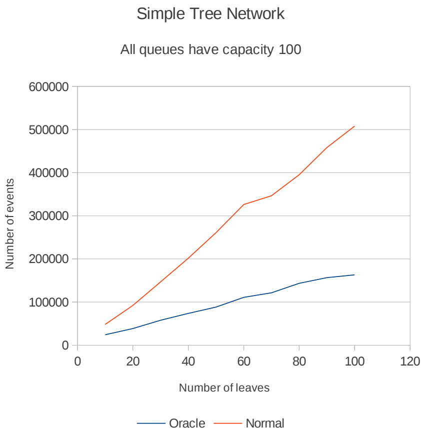

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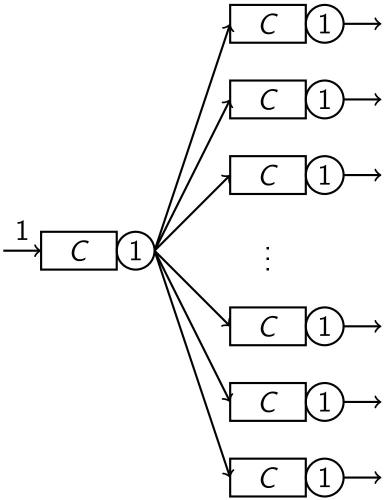

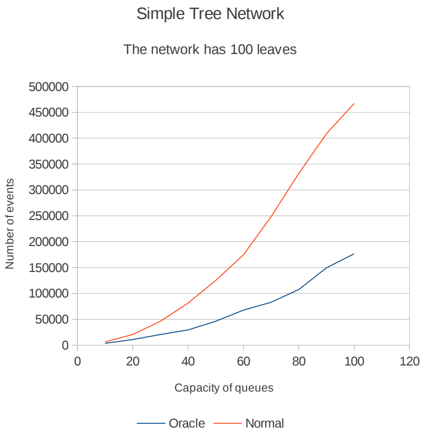

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