Perfect sampling for closed queuing networks

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Outline

Introduction

- Model
  - Closed queuing network (Gordon-Newell network or closed Jackson network)
  - Queues are ./M/1/C
  - Modeled by an ergodique markov chain with a unique stationary distribution
  - The stationary distribution do not have a product form
- Perfect sampling
  - Coupling from the past (Propp Wilson - 1996)
  - Sample stationary distribution

Presentation of the model

We consider a closed queuing network with:

- K queues
- M customers

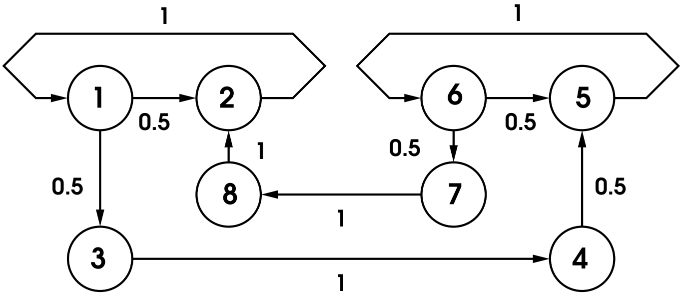
A customer who is served in queue i is routed to queue j with probability $p_{i,j}$

The topology of the network is a directed graph $G = (Q, R)$:

- Q the set of queues
- $R = \{(i, j) \ : \ p_{i,j} > 0\}$
- G is strongly connected

Toy example - The network

- $K = 8$ queues
- G strongly connected



The queue k

- Each queue $k \in \{1, 2, \ldots, K\}$ has:
  - exponential service rate μ_k
  - C_k for capacity
- The constraint for queue k :
  - The customer number x_k : $0 \leq x_k \leq C_k$
  - A customer can not be routed in queue k if the queue is full.

State space

- The state space:

$$\mathcal{S} = \{x = (x_1, x_2, \ldots, x_K) \in \mathbb{N}^K \mid \sum_{k=1}^{K} x_k = M, \ \forall k \ 0 \leq x_k \leq C_k\}$$

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- If $C = (M, M, \ldots, M)$ then $|\mathcal{S}| = \binom{K+M-1}{K-1} \leq M^K$

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- $|\mathcal{S}| \leq \binom{K+M-1}{K-1}$

Toy example - State space

- Parameters :
  - $K = 8$ queues
  - $M = 4$ customers
  - Capacity $C = (3, 3, 3, 3, 3, 3, 3, 3)$
- $\mathcal{S} = \{31000000, 01200010, 01100002, 00101101, 20000200, 11000200, 20001001, 02000200, 10002100, 10010110, 00300010, 02001001, 01002100, 01010110, 10100200, 00012010, 01100200, 00020020, 10101001, 30000100, 21000100, 01101001, 00102100, 00110110, 12000100, 00010102, 20002000, 03000100, \ldots\}$
- $|\mathcal{S}| = 322$

Transitions

- Let $(i, j) \in R$
- The vector $e_i \in \{0,1\}^K$

$$e_i(c) = \begin{cases} 1 & \text{if } i = c \\ 0 & \text{else.} \end{cases}$$

- The **transition function** : $t_{i,j} \; : \; \mathcal{S} \to \mathcal{S}$

$$t_{i,j}(x) = \begin{cases} x & \text{if } x_i = 0 \text{ or } x_j = C_j, \\ x - e_i + e_j & \text{else.} \end{cases}$$

Toy example - Transition $t_{1,3}$

- Example : transition $t_{1,3}$

States :

01100002
01300000
20001001
10300000
20101000
00111100
10210000

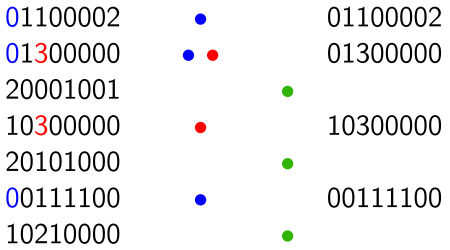
Toy example - Transition $t_{1,3}$

x	$x_1 = 0$, $x_3 = C_3$
01100002	•
01300000	• •
20001001	
10300000	•
20101000	
00111100	•
10210000	

Toy example - Transition $t_{1,3}$

x $\quad x_1 = 0$

$x_3 = C_3$



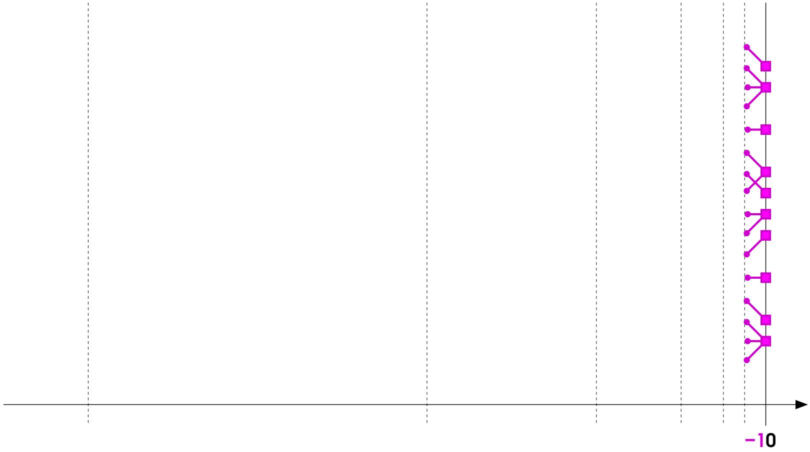
Toy example - Transition $t_{1,3}$

x	$x_1 = 0$ / $x_3 = C_3$		$t_{1,3}(x)$
01100002	• (blue)		01100002
01300000	• (blue) • (red)		01300000
20001001		• (green)	10101001
10300000	• (red)		10300000
20101000		• (green)	10201000
00111100	• (blue)		00111100
10210000		• (green)	00310000

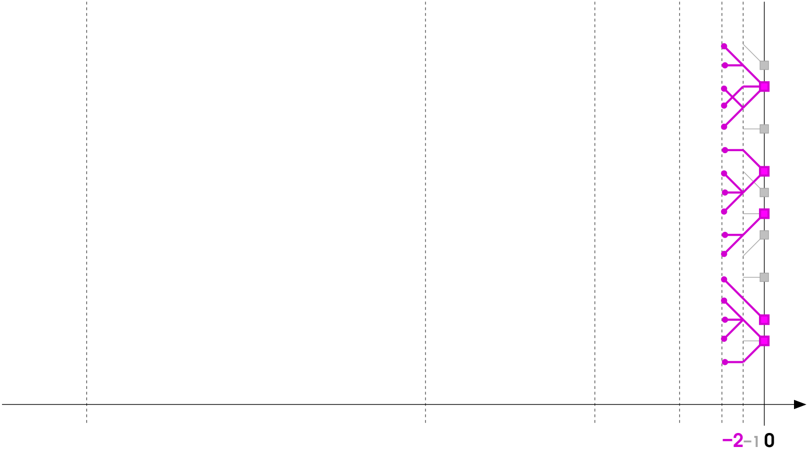
Markov chain

- The evolution of this system can be modeled by an ergodic Markov Chain:
  - $\mathcal{S}$ is the state space
  - Transition : $X_{n+1} = t_{i,j}(X_n)$ with probability $\frac{\mu_i}{\sum_{k=1}^{K} \mu_k} p_{i,j}$
- This Markov Chain has a unique stationary distribution $\mathcal{D}ist$
- **Goal** : sampling $\mathcal{D}ist$ with the perfect sampling technique

Perfect sampling



Perfect sampling



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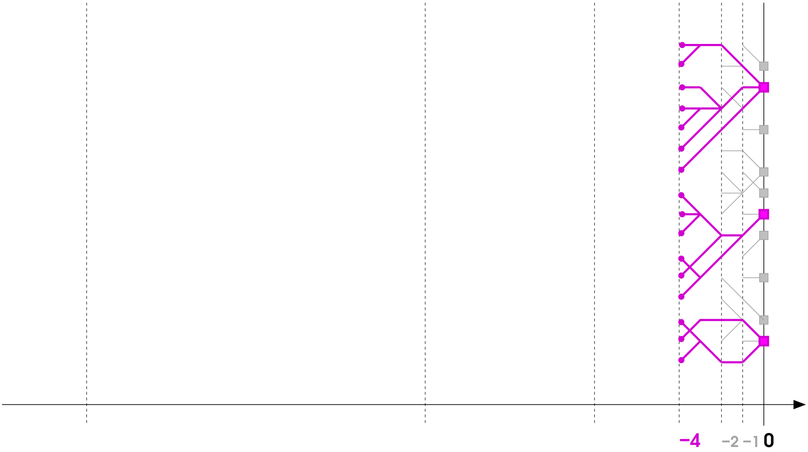

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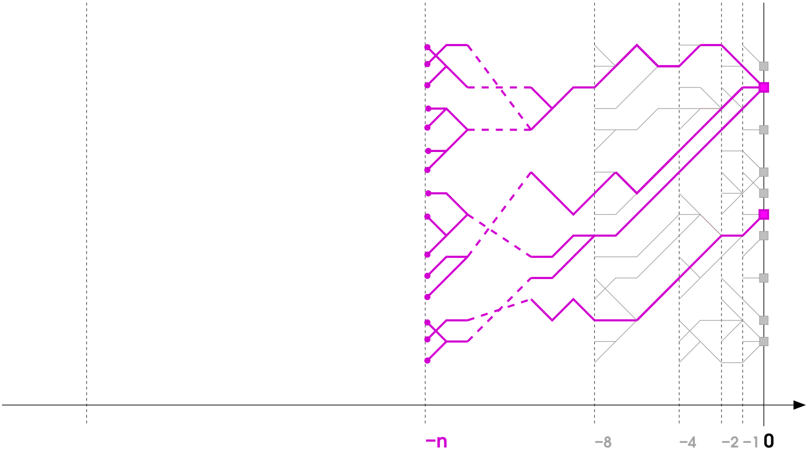

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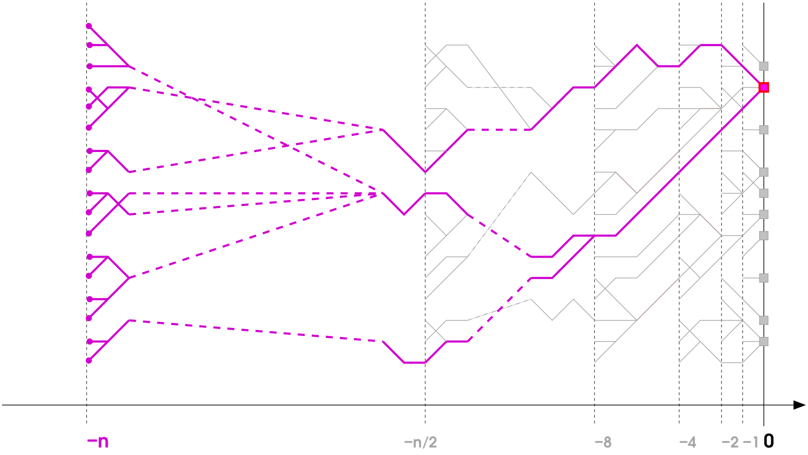


Perfect sampling for $\mathcal{D}ist$

Data: $(U_{-n} = (i_{-n}, j_{-n}))_{n \in \mathbb{N}}$ i.i.d sequence of r.v.
Result: $x \in \mathcal{S}$
begin
    $n \leftarrow 1$;
    $t \leftarrow t_{U_{-1}}$;
    while $|t(\mathcal{S})| \neq 1$ **do**
        $n \leftarrow 2n$;
        $t \leftarrow t_{U_{-1}} \circ \cdots \circ t_{U_{-n}}$;
    end
    return $\Pi(t(\mathcal{S})) = \{x_0\}$
end

- $|\mathcal{S}| \leq \binom{K+M-1}{K-1}$
- We can compute the algorithm on our toy example
- But if we choose a slightly larger K or M ...
- Problem : Data storage of $\mathcal{S}$

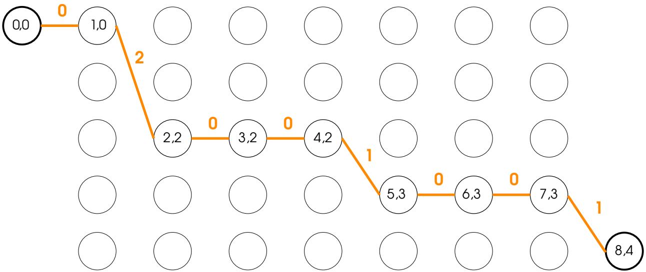
Idea

Idea

- Use the constraints of the state space
  - $\sum_{k=1}^{K} x_k = M$
  - $\forall k \; 0 \leq x_k \leq C_k$
- Represent the state space as paths in a diagram
- Make the transitions on the diagram

Toy example - Diagram idea

- Parameters :
  - $K = 8$ queues, $M = 4$ customers
  - Capacity $C = (3, 3, 3, 3, 3, 3, 3, 3)$
- States :
  - $x_1 = 02001001$
- Paths



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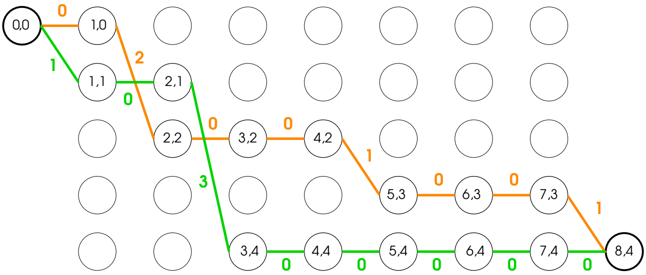
  - $K = 8$ queues, $M = 4$ customers
  - Capacity $C = (3, 3, 3, 3, 3, 3, 3, 3)$
- States :
  - $x_1 = 02001001$
  - $x_2 = 10300000$
- Paths



Complete diagram

Definition

*The **complete diagram** is a directed graph $\mathcal{D} = (N, \mathcal{A})$:*

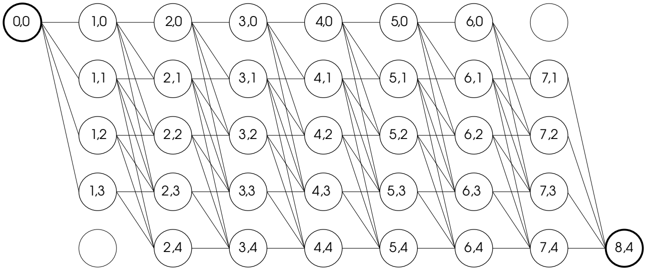
- $N = \{(c, \ell),\ \forall c \in \{1, \dots, K-1\},\ \forall \ell \in \{0, \dots, M\}\} \cup \{(0,0)\} \cup \{(K, M)\}$.
- $\mathcal{A} = \{((c-1, \ell), (c, \ell'))\ 0 \leq \ell' - \ell \leq C_c\}$.
- $(0,0)$ *is the only source node and* (K, M) *the only sink node.*

- $|A| = O(KM^2)$
- Graphically, the nodes are placed on a grid of size $(K+1) \times (M+1)$

Toy example - Complete diagram

State space and Complete diagram

$K = 8$ queues	9 columns
$M = 4$ customers	5 rows
$C = (3,3,3,3,3,3,3,3)$	$0 \leq \lvert\text{slopes}\rvert \leq 3$



Toy example - State space vs. Complete diagram

State space vs. Complete diagram

$K = 8$ queues	9 columns
$M = 4$ customers	5 rows
$C = (3,3,3,3,3,3,3,3)$	$0 \leq \lvert\text{slopes}\rvert \leq 3$
$\lvert\mathcal{S}\rvert = 322$ states	104 edges

Toy example - State space vs. Complete diagram

State space vs. Complete diagram

$K = 8$ queues	9 columns
$M = 10$ customers	11 rows
$C = (3, 3, 3, 3, 3, 3, 3, 3)$	$0 \leq \lvert\text{slopes}\rvert \leq 3$
$\lvert\mathcal{S}\rvert = 6728$ states	170 edges

Toy example - State space vs. Complete diagram

State space vs. Complete diagram

$K = 8$ queues	9 columns
$M = 10$ customers	11 rows
$C = (\infty, \infty, \infty, \infty, \infty, \infty, \infty, \infty)$	$0 \leq \lvert\text{slopes}\rvert \leq 10$
$\lvert\mathcal{S}\rvert = 19448$ states	418 edges

Paths

- $\Pi(\mathcal{D})$ set of all paths from node $(0,0)$ to node (K,M).
- $f : \mathcal{S} \to \Pi(\mathcal{D})$ the function which transforms $x = (x_1, \dots, x_K) \in \mathcal{S}$ into a path $w \in \Pi(\mathcal{D})$.

$$f(x) = ((0,0),(1,x_1),\dots,(c,\sum_{i=1}^{c} x_i),\dots,(K,M))$$

Lemma

f is a bijection between $\mathcal{S}$ and $\Pi(\mathcal{D})$.

$$f^{-1}(w) = (\ell_1, \ell_2 - \ell_1, \dots, \ell_c - \ell_{c-1}, \dots, M - \ell_{K-1})$$

Diagram definition

- $\mathcal{D} = (N, \mathcal{A})$ a complete diagram

Definition

Let $A \subseteq_{\pi} \mathcal{A}$, a ***diagram*** *$D = (N, A)$ is a subdiagram of $\mathcal{D} = (N, \mathcal{A})$*

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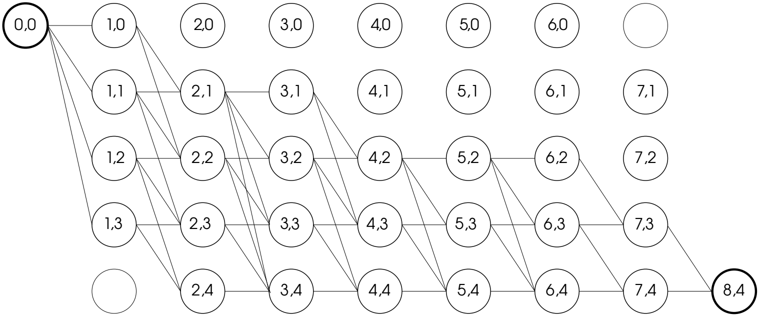
Definition

*Let $A \subseteq_\pi \mathcal{A}$, a **diagram** $D = (N, A)$ is a subdiagram of $\mathcal{D} = (N, \mathcal{A})$*

- $\Pi(D)$ the set of all paths from node $(0,0)$ to node (K, M).

Toy Example - Diagram



Representative and super-representative

A **diagram** can be represented by the set of its paths, it is also denoted $D = (N, A_\Pi)$.

- $S \subseteq \mathcal{S}$
- $\phi(S) = (N, A_{f(S)})$ *Minimal diagram containing* S
- $\psi(D) = f^{-1}(\Pi(D))$ *States on diagram* D

Definition

- *If* $S = \psi(D)$ *then* D *is a* ***representative*** *of* S.
- *If* $S \subseteq \psi(D)$ *then* D *is a* ***super-representative*** *of* S.

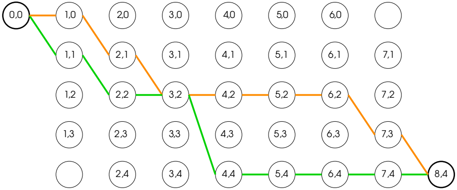
The complete diagram $\mathcal{D}$ is a representative of $\mathcal{S}$.

Example

- S

01100011
11020000

- $D = \phi(S)$



- $\psi(D)$

01100011
11020000
01120000
11000011

Transition on the diagram

- Let $(i, j) \in R$, the transition function on the diagram $T_{i,j}(D)$:

$$T_{i,j}(D) = \phi \circ t_{i,j} \circ \psi(D)$$

Proposition

(i) *If D is a super-representative of S then $T_{i,j}(D)$ is a super-representative of $t_{i,j}(S)$*

(ii) *If $|\psi(D)| = 1$ then $|\psi \circ T_{i,j}(D)| = 1$*

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- Find an algorithm to compute $T_{i,j}$ efficiently

Edges subset

- Let $D = (N, A)$ a diagram and let $(i, j) \in R$:
- $A_i \subset A$ the set of edges in the colums i
- $A_{i,j} \subset A$ the set of edges between the colums i and j
- $\overline{A_{i,j}}$ the complementary set of $A_{i,j}$ in A

Edges subset

- We order $A_{i,j}$ in 3 subsets:
  - $\mathcal{E}mpty = \{a \in A_{i,j} \mid a \in w \in \Pi(D) \text{ and } v(w_i) = 0\}$;
  - $\mathcal{F}ull = \{a \in A_{i,j} \mid a \in w \in \Pi(D) \text{ and } v(w_j) = C_j\}$;
  - $\mathcal{T}ransit = \{a \in A_{i,j} \mid a \in w \in \Pi(D) \text{ and } v(w_i) > 0 \text{ and } v(w_j) < C_j\}$.
- $A_{i,j} = \mathcal{E}mpty \cup \mathcal{F}ull \cup \mathcal{T}ransit$
- The subsets are not necessarily disjoint

Transition algorithm

- Transition algorithm $T_{i,j}$ in a diagram:

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  1. Determine the subsets : $\mathcal{E}mpty$, $\mathcal{F}ull$, $\mathcal{T}ransit$

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  2. Compute $\mathcal{T}ransit'$
     Let $a = (\ell, \ell') \in \mathcal{T}ransit$:
     - If $a \in A_i$ then $a' = (\ell, \ell' - 1)$
     - If $a \in A_{i+1,j-1}$ then $a' = (\ell - 1, \ell' - 1)$
     - If $a \in A_j$ then $a' = (\ell - 1, \ell')$

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  3. Compute ${A_{i,j}}' = \mathcal{E}mpty \cup \mathcal{F}ull \cup \mathcal{T}ransit'$

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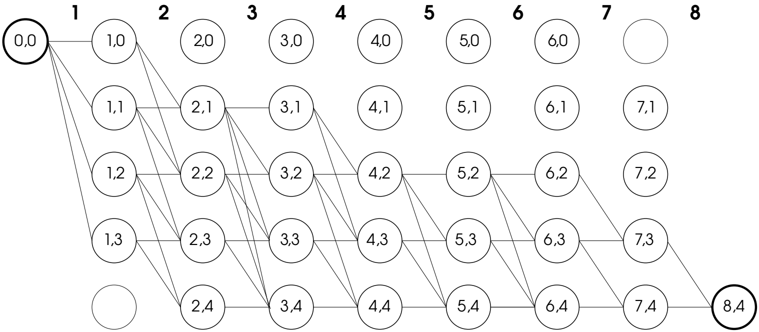
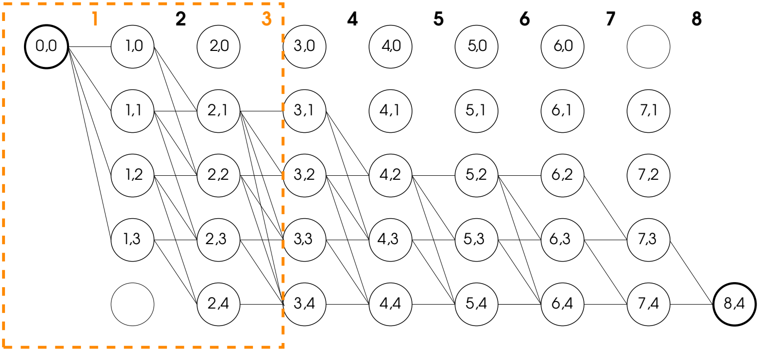
  3. Compute ${A_{i,j}}' = \mathcal{E}mpty \cup \mathcal{F}ull \cup \mathcal{T}ransit'$
  4. Return $T_{i,j}(D) = (N, {A_{i,j}}' \cup \overline{A_{i,j}})$

Compute $T_{1,3}(D)$

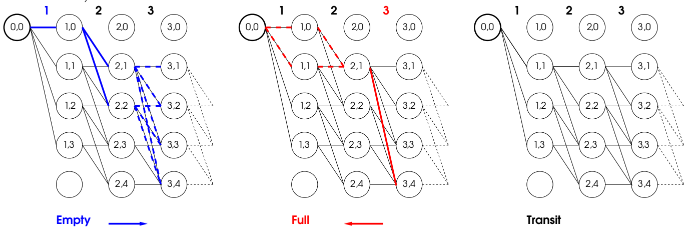


$A_{1,3}$


1
2
3
4
5
6
7
8
0,0
1,0
2,0
3,0
4,0
5,0
6,0
1,1
2,1
3,1
4,1
5,1
6,1
7,1
1,2
2,2
3,2
4,2
5,2
6,2
7,2
1,3
2,3
3,3
4,3
5,3
6,3
7,3
2,4
3,4
4,4
5,4
6,4
7,4
8,4

Step 1 : Determine the subsets

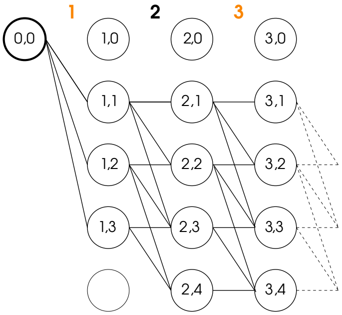
- Set $A_{1,3}$



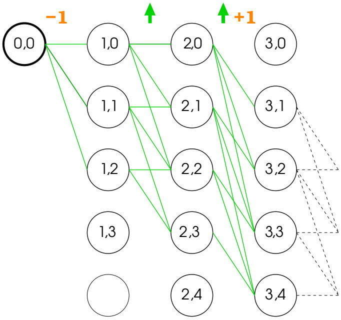
- States representated

010. . .	110. . .	021. . .	200. . .	103. . .
011. . .	111. . .	022. . .	201. . .	220. . .
012. . .	112. . .	100. . .	202. . .	220. . .
013. . .	120. . .	101. . .	210. . .	300. . .
020. . .	121. . .	102. . .	211. . .	

Step 2 : Compute *Transit'*



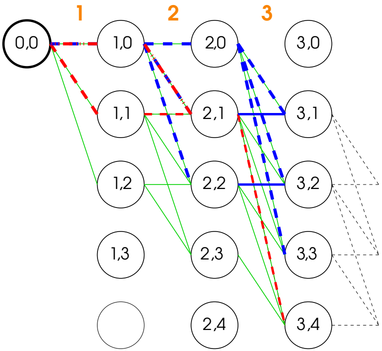
Transit



Transit'

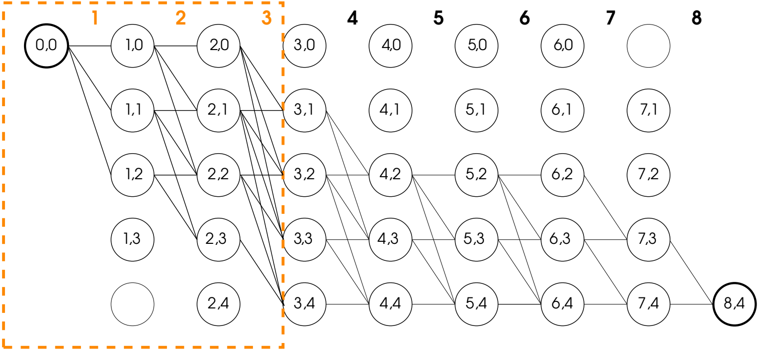
Step 3 : Compute

$$A' = \mathcal{E}mpty \cup \mathcal{F}ull \cup \mathcal{T}ransit'$$



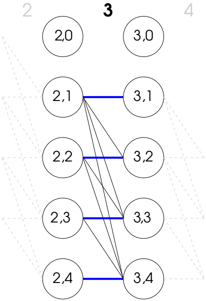
Step 4 : Return

$$T_{i,j}(D) = (N, {A_{i,j}}' \cup \overline{A_{i,j}})$$



Implementation issues & complexity

- Use K binaries matrices $M+1 \times M+1$ to represent a diagram :
  - Each column is represented by an incidence matrix
  - Represent the subsets as submatrices



	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	0	1	1	1
3	0	0	0	1	1
4	0	0	0	0	1

- Complexity for one transition is in $O(KM^2)$

Coupling on diagram ?

- Does exist a finite sequence of transitions $T = T_{i_\rho,j_\rho} \circ \cdots \circ T_{i_1,j_1}$ such that $|\Pi(T(\mathcal{D}))| = 1$?

Proof idea - Coupling on $\mathcal{S}$

- $t = t_{i_\rho, j_\rho} \circ \cdots \circ t_{i_1, j_1}$ such that $|t(\mathcal{S})| = 1$?

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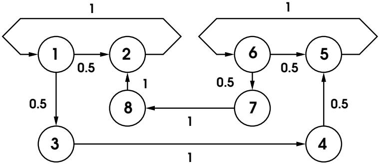

- $M = 5$ customers, $C_k = 2$ in each queue
- $\exists t$ such that $\forall x \in \mathcal{S}$

$$t(x) = (0, 0, 0, 0, 0, 1, 2, 2)$$

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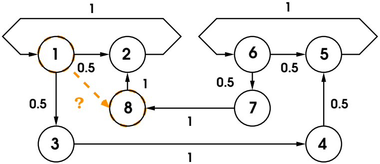

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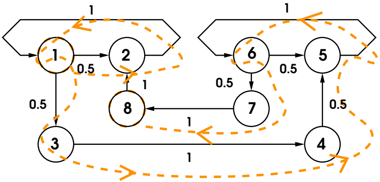

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$$t(x) = (0, 0, 0, 0, 0, 1, 2, 2)$$

- Path in network : w=(1,2,1,3,4,5,6,7,8)

Proof idea - Coupling on $\mathcal{S}$

- $M = 5$ customers, $C_k = 2$ in each queue
- $t_{1,2},\ t_{2,1},\ t_{1,3},\ t_{3,4},\ \dots,\ t_{7,8}$

w=(1,2,1,3,4,5,6,7,8)

x

01101002
20101001
10011101
20101100

- $\forall x \in \mathcal{S} \quad t(x) = (0,0,0,0,0,1,2,2)$

Proof idea - Coupling on $\mathcal{S}$

- $M = 5$ customers, $C_k = 2$ in each queue
- $t_{1,2},\ t_{2,1},\ t_{1,3},\ t_{3,4},\ \ldots,\ t_{7,8}$

w=(1,2,1,3,4,5,6,7,8)

x	$t^2_{1,2}(x)$
01101002	01101002
20101001	02101001
10011101	01011101
20101100	02101100

- $\forall x \in \mathcal{S} \quad t(x) = (0,0,0,0,0,1,2,2)$

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- $t_{1,2},\ t_{2,1},\ t_{1,3},\ t_{3,4},\ \dots,\ t_{7,8}$

w=(1,2,1,3,4,5,6,7,8)

x	$t^2_{1,2}(x)$	$t^2_{2,1}(x)$
01101002	01101002	10101002
20101001	02101001	20101001
10011101	01011101	10011101
20101100	02101100	20101100

- $\forall x \in \mathcal{S} \quad t(x) = (0,0,0,0,0,1,2,2)$

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- $t_{1,2},\ t_{2,1},\ t_{1,3},\ t_{3,4},\ \dots,\ t_{7,8}$

w=(1,2,1,3,4,5,6,7,8)

x	$t^2_{1,2}(x)$	$t^2_{2,1}(x)$	$t^2_{1,3}(x)$
01101002	01101002	10101002	00201002
20101001	02101001	20101001	10201001
10011101	01011101	10011101	00111101
20101100	02101100	20101100	10201100

- $\forall x \in \mathcal{S} \quad t(x) = (0,0,0,0,0,1,2,2)$

Proof idea - Coupling on $\mathcal{S}$

- $M = 5$ customers, $C_k = 2$ in each queue
- $t_{1,2},\ t_{2,1},\ t_{1,3},\ t_{3,4},\ \dots,\ t_{7,8}$

w=(1,2,1,3,4,5,6,7,8)

x	$t^2_{1,2}(x)$	$t^2_{2,1}(x)$	$t^2_{1,3}(x)$	$t^2_{3,4}(x)$
01101002	01101002	10101002	00201002	00021002
20101001	02101001	20101001	10201001	10021001
10011101	01011101	10011101	00111101	00021101
20101100	02101100	20101100	10201100	10021100

- $\forall x \in \mathcal{S} \quad t(x) = (0,0,0,0,0,1,2,2)$

Proof idea - Coupling on $\mathcal{S}$

- $M = 5$ customers, $C_k = 2$ in each queue
- $t_{1,2},\ t_{2,1},\ t_{1,3},\ t_{3,4},\ \ldots,\ t_{7,8}$

w=(1,2,1,3,4,5,6,7,8)

x	$t^2_{1,2}(x)$	$t^2_{2,1}(x)$	$t^2_{1,3}(x)$	$t^2_{3,4}(x)$
01101002	01101002	10101002	00201002	00021002
20101001	02101001	20101001	10201001	10021001
10011101	01011101	10011101	00111101	00021101
20101100	02101100	20101100	10201100	10021100

- $\forall x \in \mathcal{S} \quad t(x) = (0,0,0,0,0,1,2,2)$

Proof idea - Coupling on $\mathcal{S}$

- $M = 5$ customers, $C_k = 2$ in each queue
- $t_{1,2},\ t_{2,1},\ t_{1,3},\ t_{3,4},\ \dots,\ t_{7,8}$

w=(1,2,1,3,4,5,6,7,8)

x	$t_{1,2}^2(x)$	$t_{2,1}^2(x)$	$t_{1,3}^2(x)$	$t_{3,4}^2(x)$
01101002	01101002	10101002	00201002	00021002
20101001	02101001	20101001	10201001	10021001
10011101	01011101	10011101	00111101	00021101
20101100	02101100	20101100	10201100	10021100

- $\left(t = {t_{7,8}}^{C_8} \circ \cdots \circ {t_{1,3}}^{C_3} \circ {t_{2,1}}^{C_1} \circ {t_{1,2}}^{C_2}\right)^M$
- $\forall x \in \mathcal{S} \quad t(x) = (0,0,0,0,0,1,2,2)$

Proof idea - Representated state

- State space : $M = 5,\ C_k = 2$

w	1	2	1	3	4	5	6	7	8
	0	0	0	0	0	0	0	0	0
	2	2	2	2	2	2	2	2	2

Proof idea - Representated state

- State space : $M = 5, \ C_k = 2$

w	1	2	1	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
4	2	2	2	2	2	2	2	2	2

- Representated states for $w_3 = (1, 2, 1)$

0	1	2	3	4
00...	10...	20...	21...	22...
	01...	02...	12...	
		11...		

Proof idea - Saturated state

- State space : $M = 5,\ C_k = 2$

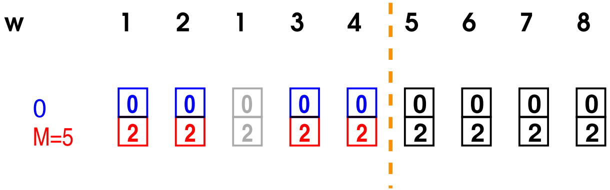
w	1	2	1	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
4	2	2	2	2	2	2	2	2	2

- Saturated states for $w_3 = (1, 2, 1)$

0	1	2	3	4
00...	10...	20...	21...	22...
	01...	02...	12...	
		11...		

Proof idea - Saturated state

- State space : $M = 5,\ C_k = 2$



- Saturated states for $w_5 = (1, 2, 1, 3, 4)$

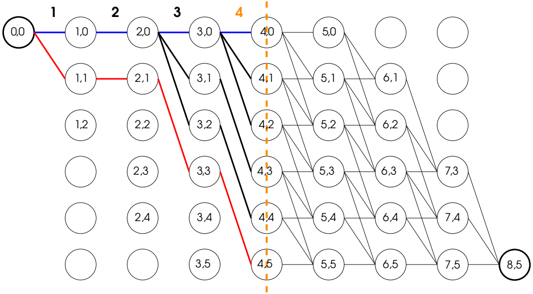
0	1	2	3	4	M=5
0000...	0001...	0002...	0012...	0022...	1022...

Proof idea - Saturated diagram

- Saturated states $\mathcal{S}_5$ for $w_5 = (1, 2, 1, 3, 4)$

0	1	2	3	4	M=5
0000...	0001...	0002...	0012...	0022...	1022...

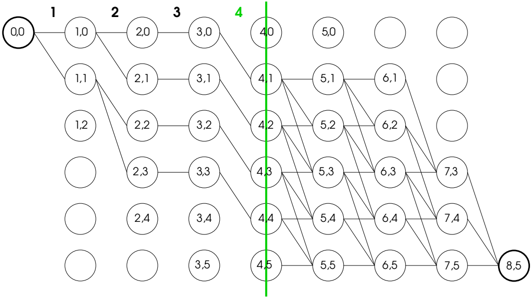
- Saturated diagram $\mathcal{D}_5$



- Bijection with the restriction w_5

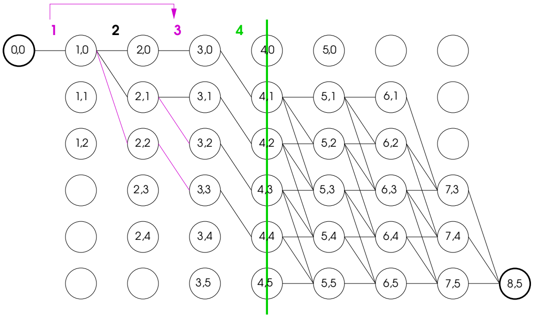
Proof idea - Tree property

- Tree property $\mathcal{T}_4$



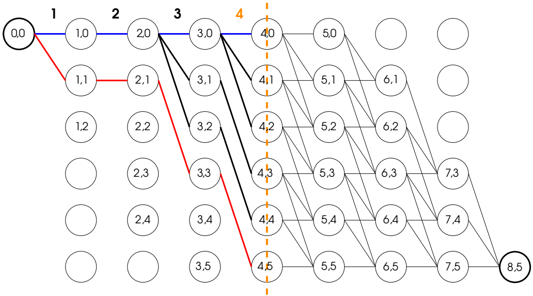
Proof idea - Tree property

- Tree property $\mathcal{T}_4$



Proof idea - Tree property

- Tree property $\mathcal{T}_4$



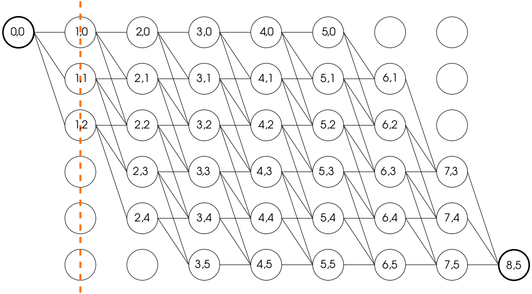
Lemma

- *Diagram $\mathcal{D}_p$ satisfies property $\mathcal{T}_c$ where c is the largest queues number on w_p.*
- *If D_p satisfies $\mathcal{T}_c$ then $T_{i,j}(D_p)$ satisfies $\mathcal{T}_c$*

Proof idea - Transition T

Complete diagram $\mathcal{D}$ has the property $\mathcal{T}_1$ and $\mathcal{D} = \mathcal{D}_1$

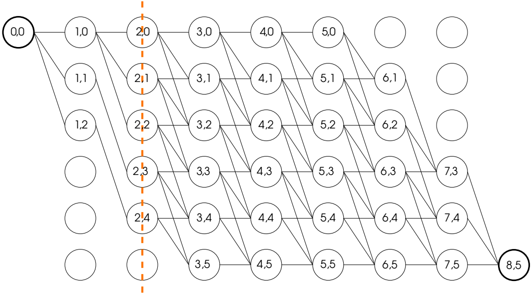
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$$T_{1,2}{}^{2}(\mathcal{D}) = \mathcal{D}_2$$

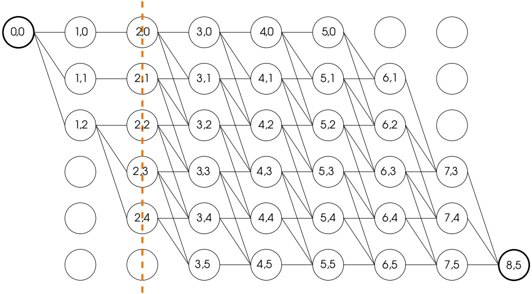
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$$T_{2,1}{}^2 \circ T_{1,2}{}^2(\mathcal{D}) = \mathcal{D}_3$$

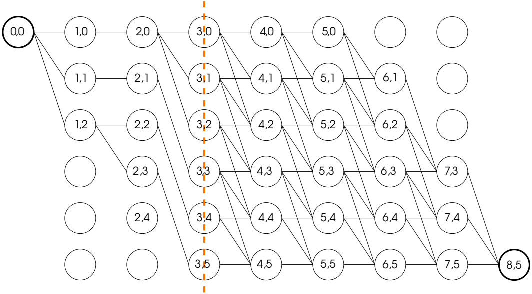
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$\exists T_4$ such that $T_4(\mathcal{D}) = \mathcal{D}_4$

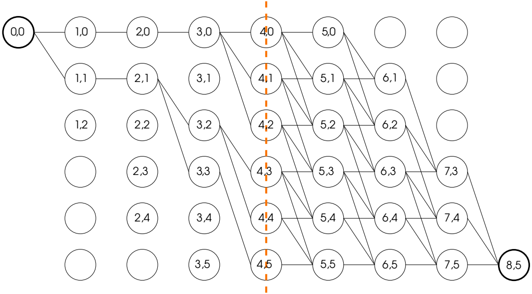
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$\exists T_5$ such that $T_5(\mathcal{D}) = \mathcal{D}_5$

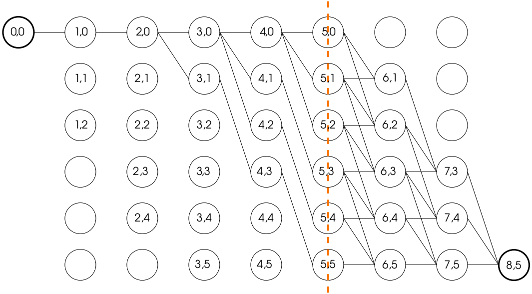
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$\exists T_6$ such that $T_6(\mathcal{D}) = \mathcal{D}_6$

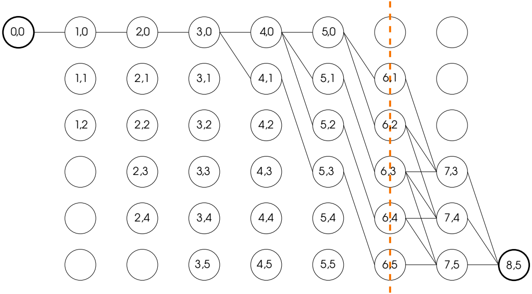
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$\exists T_7$ such that $T_7(\mathcal{D}) = \mathcal{D}_7$

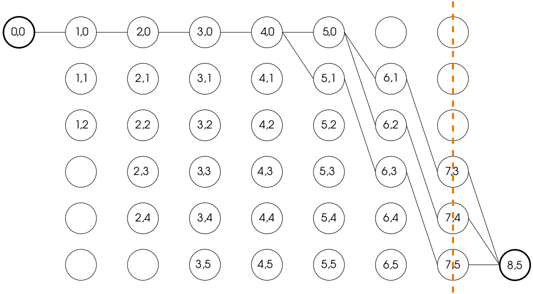
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$\exists T_8$ such that $T_8(\mathcal{D}) = \mathcal{D}_8$

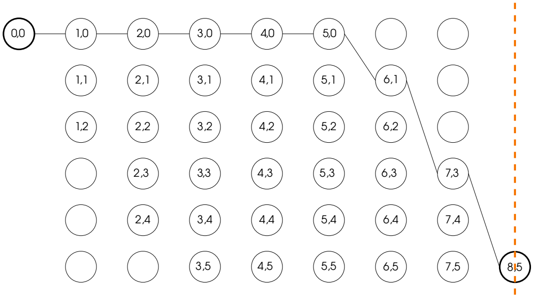
w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

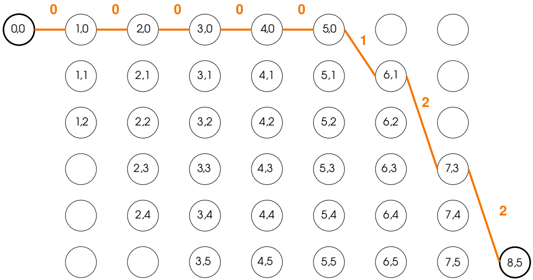
$\exists T_9$ such that $T_9(\mathcal{D}) = \mathcal{D}_9$

w=(1,2,1,3,4,5,6,7,8)



Proof idea - Transition T

$$\exists\ T = T_{i_\rho, j_\rho} \circ \cdots \circ T_{i_1, j_1} \text{ such that } |\Pi(T(\mathcal{D}))| = 1$$



Perfect sampling with $\mathcal{S}$

Data: $(U_{-n} = (i_{-n}, j_{-n}))_{n \in \mathbb{N}}$ i.i.d sequence of r.v.
Result: $x \in \mathcal{S}$
begin
  $n \leftarrow 1$;
  $t \leftarrow t_{U_{-1}}$;
  while $|t(\mathcal{S})| \neq 1$ **do**
    $n \leftarrow 2n$;
    $t \leftarrow t_{U_{-1}} \circ \cdots \circ t_{U_{-n}}$;
  end
  return $\Pi(t(\mathcal{S})) = \{x_0\}$
end

Algorithm 1: Perfect sampling with the original state space

Perfect sampling with $\mathcal{D}$

Data: $(U_{-n} = (i_{-n}, j_{-n}))_{n \in \mathbb{N}}$ i.i.d sequence of r.v.
Result: $w \in \Pi(\mathcal{D})$
begin
  $n \leftarrow 1$;
  $T \leftarrow T_{U_{-1}}$;
  while $|\Pi(T(\mathcal{D}))| \neq 1$ **do**
    $n \leftarrow 2n$;
    $T \leftarrow T_{U_{-1}} \circ \cdots \circ T_{U_{-n}}$;
  end
  return $\Pi(T(\mathcal{D})) = \{w_0\}$
end

Algorithm 2: Perfect sampling with diagram

Perfect simulation algorithm

Theorem

There exists a finite sequence of transitions $T = T_{i_\rho, j_\rho} \circ \cdots \circ T_{i_1, j_1}$ *such that* $|\Pi(T(\mathcal{D}))| = 1$.

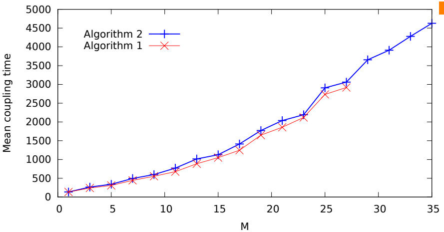
- The theorem implies the algorithm ends.
- If $(U_{-n} = (i_{-n}, j_{-n}))_{n \in \mathbb{N}}$ is the same i.i.d sequence for both algorithms then the proposition implies :
  - If $|\Pi(T(\mathcal{D}))| = 1$ then $w_0 = \{\Pi(T(\mathcal{D}))\} = \{t(\mathcal{S})\}$
  - w_0 follow the stationary distribution $\mathcal{D}ist$

Definition

The ***coupling time*** *is the n value when the algorithm ends.*

Toy example - Simulation

- Mean of coupling time



- Parameters :
  - $K = 8$
  - $M = \{1, 3, \dots, 30\}$ and $C_k = \frac{M}{2}$
  - One point : 100 simulations

- Results :
  - Ratio between algorithms 1 and 2 : $1 \leq r \leq 1.2$
  - When $M > 27$ algorithm 1 does not work

Conclusion

- Contribution : a new efficient perfect sampling technique on closed queuing networks
- Future work :
  - Extend the technique to other types of closed networks
  - Investigate about the coupling times
  - Improve complexity $T_{i,j}(D)$ in $O(KM)$
  - Improve the ratio

The end ...